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STRICT ERDŐS-KO-RADO FOR SIMPLICIAL COMPLEXES

(EXTENDED ABSTRACT)

Denys Bulavka^{*} Russ Woodroofe[†]

Abstract

We show that the strict Erdős-Ko-Rado property holds for sequentially Cohen-Macaulay near-cones. In particular, this implies that chordal graphs with at least one isolated vertex satisfy the strict Erdős-Ko-Rado property.

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1 Introduction

What is the largest cardinality of a family of pairwise-intersecting sets? A now-classic result of Erdős, Ko, and Rado answers this question if the sets all have the same number of elements, and are otherwise unrestricted.

Theorem 1.1 (Erdős, Ko, and Rado [4]). Let $r \leq n/2$. If \mathcal{F} is a family of pairwiseintersecting subsets of [n], each with r elements, then $|\mathcal{F}| \leq {n-1 \choose r-1}$.

If $|\mathcal{F}|$ achieves the upper bound and r < n/2, then \mathcal{F} consists of all the r-element subsets containing some fixed element.

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[†]Univerza na Primorskem, Glagoljäka 8, 6000 Koper, Slovenia. Supported in part by the Slovenian Research Agency (research program P1-0285 and research projects J1-9108, N1-0160, J1-2451). Email: russ.woodroofe@famnit.upr.si. https://osebje.famnit.upr.si/~russ.woodroofe/.

That is, under the above hypotheses a family of pairwise-intersecting objects of maximal size is given by a family with a common intersection. Moreover, under slightly stronger hypotheses, this is the only such family. Hilton and Milner [8] later gave upper bounds for pairwise-intersecting families that do not all contain a common element.

There are a large number of generalizations of Theorem 1.1. We focus on one in particular. Holroyd and Johnson asked at the 1997 British Combinatorial Conference [13] about whether an analogue of Erdős-Ko-Rado property holds for independent sets in cyclic and similar graphs. Talbot showed the answer to be "yes" in a strong sense.

Theorem 1.2 (Talbot [22]). Let n, k, r be positive integers such that $r \leq n/(k+1)$. Let G be the graph with vertex set \mathbb{Z}_n and edges consisting of those x, y such that $x - y \in \{1, \ldots, k\}$.

If \mathcal{F} is a family of pairwise-intersecting independent sets of G, each with r elements, then $|\mathcal{F}|$ is smaller than the family \mathcal{B} of all independent sets with r elements containing 0. If $|\mathcal{F}|$ achieves the upper bound and $n \neq 2r + 2$, then \mathcal{F} is \mathcal{B} up to relabeling the vertices.

Holroyd and Talbot asked whether similar results hold for independent sets in other graphs G. There are counterexamples for r around the size of a maximum independent set, but not for somewhat smaller r. Since the collection of independent sets form a simplicial complex, and since it is our main object of study, we introduce it now. A simplicial complex K with vertex set V is a set system $K \subseteq 2^V$ that is closed under taking subset, i.e., if $F \in K$ and $G \subseteq F$ then $G \in K$. In this article we will assume that the vertex set V is equipped with a total order and consequently we can identify it with $[|V|] = \{1, \ldots, |V|\}$ respecting this order. The members of K are called *faces*, the faces of size r are called r-faces, the number of r-faces in K is denoted by $f_r(K)$ and the maximal faces with respect to inclusion are called *facets*. The *dimension of a face* F is defined by $\dim(F) = |F| - 1$ and the dimension of K by $\dim(K) = \max_{F \in K} \dim(F)$. The r-skeleton of K, denote by $K^{(r)}$, is the set of faces from K of dimension at most r. The pure rskeleton of K is the simplicial complex given by all the faces from K of dimension r and their subsets. Given a face $F \in K$, the link of F in K is defined as the simplicial complex $lk(F, K) = \{G \in K : G \cap F = \emptyset \text{ and } F \cup G \in K\}$. By $K[S] = \{F \in K : F \subseteq S\}$ we denote the *induced* simplicial complex on the vertices S. For two simplicial complexes K and L, the *join* is defined by $K * L = \{F \sqcup G : F \in K, G \in L\}$, where \sqcup denotes disjoint union.

Conjecture 1.3 (Holroyd and Talbot [10], extended by Borg to arbitrary simplicial complexes [1]). Let K be a simplicial complex whose smallest facet has d vertices, and let $r \leq d/2$. If \mathcal{F} is a family of pairwise-intersecting faces of K, each with r elements, then there is some vertex v of K so that $|\mathcal{F}| \leq f_{r-1}(\operatorname{lk}(v, K))$. If r < d/2 and $|\mathcal{F}|$ achieves the upper bound, then \mathcal{F} consists of the faces containing some vertex v.

If a simplicial complex K satisfies the upper bound of Conjecture 1.3 at a specified value of r, then we say that K is r-EKR. If every intersecting family of maximum size has a common intersection, then we say that K is *strictly* r-EKR. We abuse terminology to say that a graph is (strictly) r-EKR if its independence complex has the same property.

There has been considerable work on Conjecture 1.3. Hurlbert and Kamat showed [11] that any chordal graph with an isolated vertex satisfies the upper bound of Conjecture 1.3.

Borg showed [1] that the conjecture holds asymptotically, in the precise sense that if the minimal facet cardinality of a simplicial complex K is at least $(r-1)\binom{3r-3}{2} + r$, then K satisfies r-EKR. Other related works are [10, 9, 19]. Rather than working with arbitrary simplicial complexes we will focus on the so called sequentially Cohen-Macaulay near-cones which we introduce now. A simplicial complex K is a near-cone with apex v if for every $F \in K$ and every $w \in F$ we have that $F \setminus \{w\} \cup \{v\} \in K$. A simplicial complex is called Cohen-Macaulay over \mathbb{F} if for every face $F \in K$ we have that $H_i(\operatorname{lk}(F,K),\mathbb{F}) = 0$ for $i < \dim(\operatorname{lk}(F, K))$, that is the reduced homology of every link vanishes on every dimension except possibly the top one. A simplicial complex is said to be sequentially Cohen-Macaulay over \mathbb{F} if for every r, the pure r-skeleton of K is Cohen-Macaulay over \mathbb{F} . From now on we will assume that the field has characteristic 0 and drop it from the notation. The second author showed more generally [25] that any sequentially Cohen-Macaulay nearcone satisfies the upper bound of Conjecture 1.3. We note here that the independence complex of a graph G is a cone if and only if G has an isolated vertex. Moreover, the class of sequentially Cohen-Macaulay simplicial complexes is a broad class that includes the independence complexes of chordal graphs and many others [2, 17, 24]. Neither Hurlbert and Kamat nor the second author addressed the strict r-EKR property.

The main purpose of the current paper is to fill in this gap. We show:

Theorem 1.4. Let $2 \le r < d/2$. If the simplicial complex K is a sequentially Cohen-Macaulay near-cone with minimal facet cardinality d, then K is strictly r-EKR, that is the pairwise-intersecting families of maximum size consist of all r-faces containing an apex vertex.

The novelty of our techniques is to combine algebraic and combinatorial shifting operations. We also make use of some of the ideas behind proofs of the Hilton-Milner theorem [7, 12].

This article is organized as follows, in Section 2 we review the main results needed for the proof. In Section 3 we give the proof of Theorem 1.4.

2 Shifting

A set system $\mathcal{F} \subseteq {\binom{[n]}{r}}$ is said to be *shifted* if for every $F \in \mathcal{F}$ and $i, j \in [n]$ such that $i < j, j \in F$ and $i \notin F$ we have that $F \setminus \{j\} \cup \{i\} \in \mathcal{F}$. In this section we review two operations that assign to a set system another set system that is shifted while preserving several properties of interest.

Given a set system $\mathcal{F} \subseteq {\binom{[n]}{r}}$, and $F \in \mathcal{F}$. Let $i, j \in [n]$ such that i < j, the combinatorial shift Shift_{i,j} is defined by

Shift_{*i*,*j*}(*F*) =

$$\begin{cases}
F \setminus \{j\} \cup \{i\} & \text{if } j \in F, i \notin F \text{ and } F \setminus \{j\} \cup \{i\} \notin \mathcal{F}, \\
F & \text{otherwise.}
\end{cases}$$

$$\operatorname{Shift}_{i,j}(\mathcal{F}) = \{\operatorname{Shift}_{i,j}(F) \colon F \in \mathcal{F}\}.$$

We will be using the following properties of combinatorial shifting [6].

Theorem 2.1. Let $\mathcal{F} \subseteq {\binom{[n]}{r}}$ and $i, j \in [n]$ such that i < j.

- 1. $|\operatorname{Shift}_{i,j}(\mathcal{F})| = |\mathcal{F}|.$
- 2. If $\mathcal{G} \subseteq \mathcal{F}$, then $\text{Shift}_{i,j}(\mathcal{G}) \subseteq \text{Shift}_{i,j}(\mathcal{F})$ where in each case we do the combinatorial shifting according to the respective family.
- 3. If \mathcal{F} is shifted, then $\text{Shift}_{i,j}(\mathcal{F}) = \mathcal{F}$.
- 4. If \mathcal{F} is pairwise-intersecting, then $\text{Shift}_{i,j}(\mathcal{F})$ is pairwise-intersecting.

By iterating the combinatorial shifting operation we will eventually obtain a set system that is shifted, but the final set system is dependent on the order of the shifts.

Kalai [15] introduced a shifting operation that produces a shifted set system preserving several algebraic properties, so-called *(exterior) algebraic shifting.* This operation assigns to a simplicial complex K a shifted simplicial complex $\Delta(K)$. We would like to point out that in contrast to combinatorial shifting, algebraic shifting works in one step rather than an iterative procedure. Here we merely state the properties we will be using.

Theorem 2.2. Let K be a simplicial complex.

- 1. [15, Theorem 2.1.2] $\Delta(K)$ is shifted.
- 2. [15, Theorem 2.2.7] $K \subseteq L$, then $\Delta(K) \subseteq \Delta(L)$.
- 3. [15, Theorem 2.1.1] $f_r(K) = f_r(\Delta(K))$.
- 4. [15, Theorem 4.1] K is Cohen-Macaulay then $\Delta(K)$ is Cohen-Macaulay.
- 5. [15, Theorem 6.2] If $\mathcal{F} \subseteq {\binom{[n]}{a}}$ and $\mathcal{G} \subseteq {\binom{[n]}{b}}$ are cross-intersecting, then $\Delta(\mathcal{F})$ and $\Delta(\mathcal{G})$ are cross-intersecting.
- 6. [18, Theorem 5.3] If K is a near-cone with apex v, then $\Delta(K) = (1 * \Delta(\operatorname{lk}(v, K))) \cup B$ where $B = \{F \in \Delta(K) : 1 \notin F\}$. In particular, $f_r(\operatorname{lk}(1, \Delta(K))) = f_r(\Delta(\operatorname{lk}(v, K))) = f_r(\operatorname{lk}(v, K))$.

Since the minimal facet cardinality plays a key role in Conjecture 1.3. We need to be able to control its behavior when performing (algebraic) shifting operations. For this purpose we introduce the following definition of *depth of a simplicial complex* K

depth
$$K = \max\{d: K^{(d)} \text{ is Cohen-Macaulay}\}$$
.

The depth of a simplicial complex is one less than the depth of its Stanley-Reisner ring [21].

Corollary 2.3 ([3]). The minimum facet dimension of $\Delta(K)$ is at least d if and only if $K^{(d)}$ is Cohen-Macaulay over \mathbb{F} .

From the above corollary it follows that depth K + 1 is the minimum facet cardinality of $\Delta(K)$ which is at most the minimal facet cardinality of K. Notice that when K is sequentially Cohen-Macaulay, the minimal facet cardinality of K coincides with depth K+1and consequently with the minimal facet cardinality of its algebraic shift $\Delta(K)$.

3 Proof of Theorem 1.4

First, we adapt the approach in [7, 12] for shifted simplicial complexes. For $\{i, j\} \in {[n] \choose 2}$, let swap_{*i*,*j*} denote the function exchanging vertices *i* and *j*.

Lemma 3.1. Let K be a shifted simplicial complex with minimal facet cardinality d and $\mathcal{F} \subseteq K$ a non-trivial pairwise-intersecting family of r-faces with $r \leq d/2$ of maximal size. Then, there exists a shifted non-trivial pairwise-intersecting family $\mathcal{F}' \subseteq K$ of r-faces such that $|\mathcal{F}| = |\mathcal{F}'|$.

Proof. Consider \mathcal{F} of maximal size. We apply $\operatorname{Shift}_{i,j}$ repeatedly to \mathcal{F} until it results in a trivial pairwise-intersecting family. Let $\operatorname{Shift}_{s,t}$ be the first shifting operation making the family trivial and let \mathcal{H} be the non-trivial family before applying the last shifting $\operatorname{Shift}_{s,t}$. Because K is shifted, the repeated application of combinatorial shifting to the pairwise-intersecting family keeps the family in the simplicial complex at each step. That is, $\operatorname{Shift}_{i,j}(\mathcal{F}) \subseteq \operatorname{Shift}_{i,j}(K) = K$, where the first inclusion follows from Theorem 2.1.2 and the last step from Theorem 2.1.3.

If $\operatorname{Shift}_{1,s}(\mathcal{H})$ is non-trivial and $\operatorname{Shift}_{2,t} \circ \operatorname{Shift}_{1,s}(\mathcal{H})$ is non-trivial then applying $\operatorname{Shift}_{1,2}$ to this last family gives a trivial one and we are in the same situation as [7, Proposition 1.6], that is s = 1 and t = 2. If $\operatorname{Shift}_{1,s}(\mathcal{H})$ is trivial while $\operatorname{Shift}_{2,s}(\mathcal{H})$ is non-trivial, then $\operatorname{Shift}_{1,2} \circ \operatorname{Shift}_{2,s}(\mathcal{H})$ is trivial and we are again in the same situation as above. The remaining case is when we have that $\operatorname{Shift}_{1,s}(\mathcal{H})$ and $\operatorname{Shift}_{2,s}(\mathcal{H})$ (or $\operatorname{Shift}_{1,t}(\mathcal{H})$ and $\operatorname{Shift}_{2,t}(\mathcal{H})$) are both trivial. We will use repeatedly the following argument: if \mathcal{F} is non-trivial pairwiseintersecting family of maximal size, and $\operatorname{Shift}_{i,j}(\mathcal{F})$ is trivial then $\{i, j\} \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and, because of maximality of $|\mathcal{F}|$, we have that

$$\mathcal{T}_{i,j} = \{T \cup \{i,j\} \colon T \in \operatorname{lk}(\{i,j\},K), |T| = r - 2\} \subseteq \mathcal{F}.$$

Let $3 \leq s_0 < s$ such that $\mathcal{H}' = \text{Shift}_{s_0,s}(\mathcal{H})$ is non-trivial, we take the first one if it exists, or we set $s_0 = s$ otherwise. We notice that $s_0 \leq r+1$, since otherwise every member of \mathcal{H} would contain [r+1] or s, since the first option is not possible due to its size then \mathcal{H}' is trivial, which is a contradiction. Since $\text{Shift}_{i,s_0}(\mathcal{H}')$ is trivial for all $i \in [s_0 - 1]$ while \mathcal{H}' is not, a routine computation shows that the following holds: $\mathcal{T}_{i,s_0} \subseteq \mathcal{H}'$ for $i \in [s_0 - 1]$; for each $F \in \mathcal{H}'$ we have that $[s_0 - 1] \subseteq F$ or $s_0 \in F$; for each $i \in [s_0 - 1]$ there exist $F_i \in \mathcal{H}'$ such that $i \notin F_i$ and $s_0 \in F_i$; there exist F_0 such that $[s_0 - 1] \subseteq F_0$ and $s_0 \notin F_0$. Finally, set $\mathcal{G} = \text{swap}_{1,s_0}(\mathcal{H}')$ and $G_i = \text{swap}_{1,s_0}(F_i)$, then $1 \notin G_0$ and $s_0 \notin G_1$.

Claim 1: \mathcal{G} is intersecting. The only non-trivial case to verify is if $F, F' \in \mathcal{H}'$ are such that $F \cap [s_0] = [s_0 - 1]$ and $F' \cap [s_0] = s_0$. Since \mathcal{H}' is intersecting, there exists $x \in F \cap F'$. Then, $x \notin [s_0]$ and consequently it is not affected by $\operatorname{swap}_{1,s_0}$. From this we can conclude that $x \in \operatorname{swap}_{1,s_0}(F) \bigcap \operatorname{swap}_{1,s_0}(F')$.

Claim 2: \mathcal{G} is non-trivial. If $x \in \bigcap \mathcal{G}$, then since $\bigcap \mathcal{H}' = \emptyset$ we must have that x = 1 or $x = s_0$. But, $1 \notin G_0$ and $s_0 \notin G_1$.

Claim 3: $\mathcal{G} \subseteq K$. For $F \in \mathcal{H}'$ such that $s_0 \in F$ and $1 \notin F$, $\operatorname{swap}_{1,s_0}(F) \in K$ since K is shifted. For $F \in \mathcal{H}'$ such that $1 \in F$ and $s_0 \notin F$ consider F' given by adding the first r

vertices of $[n] \setminus F$ to F, this is a face since the minimal facet cardinality is $d \ge 2r$ and it contains s_0 since $s_0 \le r+1$. Consequently $\operatorname{swap}_{1,s_0}(F) \in K$.

Claim 4: $\mathcal{T}_{1,2} \subseteq \mathcal{G}$. Since $s_0 \geq 3$, for $i \in [s_0 - 1]$ we have that $\mathcal{T}_{i,s_0} \subseteq \mathcal{H}'$ and consequently $\mathcal{T}_{1,i} = \operatorname{swap}_{1,s_0}(\mathcal{T}_{i,s_0}) \subseteq \operatorname{swap}_{1,s_0}(\mathcal{H}') = \mathcal{G}$.

Next, we apply repeatedly $\text{Shift}_{i,j}$ with $3 \leq i < j \leq n$ to \mathcal{G} until it is stable under this restricted set of combinatorial shiftings. We notice that this does not change $\mathcal{T}_{1,2}$. Denote the stable set by \mathcal{G}' .

Claim 5: \mathcal{G}' is non-trivial. Since $[s_0 - 1] \setminus \{1\} \subseteq G_0 \in \mathcal{G}$, then $[r+1] \setminus \{1\} = [s_0] \setminus 1 \cup \{s_0 + 1, \ldots, r+1\} \in \mathcal{G}'$. On the other hand, since $\mathcal{T}_{1,2} \subseteq \mathcal{G}'$ we have that $[r+1] \setminus i \in \mathcal{G}'$ for $i \in \{3, \ldots, r+1\}$. Because $1 \in G_2 \in \mathcal{G}$ while $2 \notin G_2$, we also have that $[r+1] \setminus \{2\} \in \mathcal{G}'$. That is, $\binom{[r+1]}{r} \subseteq \mathcal{G}'$.

Consequently applying $\text{Shift}_{i,j}$ for $1 \leq i < j \leq n$ to \mathcal{G}' does not create a trivial family. Finally, we apply $\text{Shift}_{i,j}$ for $1 \leq i < j \leq n$ repeatedly to \mathcal{G}' until it is stable and denote the resulting family by \mathcal{F}' .

We will need the following technical lemma.

Lemma 3.2 ([7, 12]). Let \mathcal{F} be an pairwise-intersecting shifted family. For every $F \in \mathcal{F}$ there exists $l \geq 1$ such that $|F \cap [2l-1]| \geq l$. Moreover, the maximum such l = l(F) satisfies $|F \cap [2l(F)]| = l(F)$.

The following function was previously defined in [7] in the unrestricted context. We extend it to the setting of simplicial complexes. Let K be a shifted simplicial complex with vertex set [n] and $\mathcal{F} \subseteq K$ a shifted pairwise-intersecting family of r-faces, set

$$\alpha \colon \mathcal{F} \to (\mathrm{st}(1,K) \setminus \mathrm{st}(1,K[[n] \setminus [2,r+1]])) \cup \{2,\ldots,r+1\}$$

given by

$$\alpha(F) = \begin{cases} F & \text{if } 1 \in F \text{ or } [2, r+1] \subseteq F, \\ F\Delta[2l(F)] & \text{otherwise.} \end{cases}$$

The following lemma shows that α is well defined and injective.

Lemma 3.3. For $F \in \mathcal{F}$ such that $\alpha(F) \neq F$ we have that: (1) $1 \in \alpha(F)$. (2) $\alpha(F) \notin \mathcal{F}$. (3) $\alpha(F) \cap [2, r+1] \neq \emptyset$. (4) α is injective. (5) $\alpha(F) \in K$.

Proof. Properties (1-4) were proved previously [7], we only need to verify property (5). Notice that $d/2 \ge r \ge |F \cap [2l(F)]| = l(F)$. In particular, $|F \cup [2l(F)]| = |F| + |[2l(F)]| - |F \cap [2l(F)]| \le 2r \le d$. Because K is shifted with minimal facet cardinality d we have that $F \cup [2l(F)] \in K$ since it is the smallest face, with respect to the partial order, of size r+l(F) containing F. Since $\alpha(F) = F\Delta[2l(F)] \subseteq F \cup [2l(F)]$ we conclude that $\alpha(F) \in K$. \Box

Proposition 3.4. Let K be a shifted simplicial complex with vertex set [n] and minimal facet cardinality d and $r \leq d/2$. Let $\mathcal{F} \subseteq K$ be a non-trivial intersecting family of r-faces. We have that

$$|\mathcal{F}| \le f_{r-1}(\mathrm{lk}(1,K)) - f_{r-1}(\mathrm{lk}(1,K[[n] \setminus [2,r+1]])) + 1.$$

Moreover, if $2 \leq r < d/2$ then $|\mathcal{F}| < f_{r-1}(\operatorname{lk}(1, K))$.

Proof. By Lemma 3.1 we can assume that \mathcal{F} is shifted. The first part follows from the injectivity of α . For the second part, because $2r+1 \leq d$ we have that $\{1, r+2, \ldots, 2r+1\} \setminus \{i\} \subseteq [d] \setminus [2, r+1] \in K$ for $i \in [r+2, 2r+1]$. Consequently $f_{r-1}(\operatorname{lk}(1, K[[n] \setminus [2, r+1]])) \geq r \geq 2$ and the conclusion follows. \Box

Remark 3.5. It is not hard to show that if K be a near-cone with apex v and minimal facet cardinality d, then for $r \leq d/2$ we have that $K^{(r)} \subseteq v * \text{lk}(v, K)$.

Theorem 3.6. Let K be a near-cone with apex v, then K is strict r-EKR for $r < \frac{\operatorname{depth}_{\mathbb{F}} K+1}{2}$.

Proof. Let \mathcal{F} be a non-trivial intersecting family of r-faces. Since depth_F $K \leq d$ then r < d/2. By above remark we can conclude that $\mathcal{F} \subseteq v * \operatorname{lk}(v, K)$. Set $\mathcal{F}(v) = \{F \setminus \{v\} : F \in \mathcal{F}, v \in F\}$ and $\mathcal{F}(\bar{v}) = \{F : F \in \mathcal{F}, v \notin F\}$. Then $\mathcal{F}(v), \mathcal{F}(\bar{v}) \subseteq \operatorname{lk}(v, K)$ are cross-intersecting and $\mathcal{F}(\bar{v})$ is pairwise-intersecting. By Theorem 2.2.2 and Theorem 2.2.5 we have that $\Delta(\mathcal{F}(v)), \Delta(\mathcal{F}(\bar{v})) \subseteq \Delta(\operatorname{lk}(v, K))$ are cross-intersecting and $\Delta(\mathcal{F}(\bar{v}))$ is pairwise-intersecting. By Theorem 2.2.6, $\Delta(K) = (1 * \Delta(\operatorname{lk}(v, K))) \cup B$ where $B = \{F \in \Delta(K) : 1 \notin F\}$, consequently $\mathcal{F}' \subseteq \Delta(K)$ and non-trivial. Moreover, since no member of $\Delta(\mathcal{F}(\bar{v}))$ contains vertex 1 we have that

$$|\mathcal{F}'| = |\Delta(\mathcal{F}(v))| + |\Delta(\mathcal{F}(\bar{v}))| = |\mathcal{F}(v)| + |\mathcal{F}(\bar{v})| = |\mathcal{F}|$$

where we have used Theorem 2.2.3. Since K is shifted, by Theorem 2.2.1, with minimal facet cardinality depth_F K + 1, by Proposition 3.4, we can conclude that $|\mathcal{F}| < f_{r-1}(\operatorname{lk}(1, \Delta K)) = f_{r-1}(\operatorname{lk}(v, K))$ by Theorem 2.2.6.

As a corrolary we obtain Theorem 1.4.

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