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A compactification of the set of sequences of positive real numbers with applications to limits of graphs

(Extended abstract)

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Abstract

We introduce compactification results on the set of sequences of positive real numbers: under the continuum hypothesis, one can find a totally ordered set of sequences whose elements can be used as test sequences to capture every possible asympthotic growth, perhaps along a subsequence; this behaviour mimics the statement that, in a compact set of $\mathbb{R}$, every sequence has a convergent partial subsequence. These compactification results allows us to unify two notions of convergence for graphs into a single graph-convergence notion, while retaining the property that each sequence of graphs have a convergent partial subsequence. These convergent notions are the Benjamini-Schramm convergence for bounded degree graphs, regarding the distribution of r-neighbourhoods of the vertices, and the left-convergence for dense graphs, regarding the existence, for each fixed graph F, of a limiting probability that a random mapping from F to $\{G_i\}$ is a graph homomorphism.

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1 Introduction

Recently, several authors have considered the study of (large) discrete objects by, after introducing an appropriate limit notion, draw conclusions of the sequence by studying the objects that appear as their limits. Two of the most well known examples is the study of limits of sequences of graphs (e.g. [2, 9, 4]) or of permutations (see [8]). In this work, we focuss our attention to limits of graphs.

In the area of limits of graphs, one of the problems is that the properties of the sequences of graphs are radically different depending on several parameters, one of them being the density of edges. Thus, there have appeared several convergence notions depending on different growth regimes, such as the notion of left-convergence [9, 4] that works well when the density of edges is a positive proportion of the total number (dense case) yet trivializes when the sequence is of sparse graphs (non-dense), or Benjamini-Schramm convergence [2] when the sequence is of bounded degree graphs (very sparse case). Other notions of convergence for limits of graphs have been introduced; these either generalize the previous two in several ways, or consider some strenghthening of them. As some examples we can mention: L^p convergence [5, 3], action convergence [1], log–convergence [11], convergence in fragments of logic [10], for intermediate growth [6], or local-global convergence [7].

In the following, we give compactification results on the set of sequences of positive real numbers Theorem 1 and Theorem 2 that, as far as we know, are new, and we give an application of such results to limits of graphs by considering a graph limit notion that is uniform regardless of the growth regime of the number of edges, thus generalizing both the local convergence [2] and the left-convergence [9, 4]. This notion can be seen as a brute force generalization of the one by Frenkel [6]. First let us present the compactification results.

Theorem 1. *Assume the continuum hypothesis. Let $\mathcal{R}^{>0} = \{f \mid f : \omega \to \mathbb{R}^{>0}\}$ be the set of positive real sequences. Then there exists a set $A \subset \mathcal{R}^{>0}$ such that:*

1. *For each $a, b \in A$,*
$$\lim_{n\to\infty} \frac{a(n)}{b(n)} \qquad \text{is either } 0 \text{ or } \infty.$$
2. *For each $g \in \mathcal{R}^{>0}$ and each ordered embedding $\iota : (\omega, <) \to (\omega, <)$, there exists an ordered embedding $\iota_0 : (\omega, <) \to (\omega, <)$, an $a \in A$, and a $c \in (0, \infty)$ such that*
$$\lim_{n\to\infty} \frac{a(\iota(\iota_0(n)))}{g(\iota(\iota_0(n)))} = c$$

Theorem 1 claims to obtain a totally ordered set A (ordered with the relation $a < b \iff \lim_{n\to\infty} \frac{a(n)}{b(n)} = 0$) with the property that for any partial sequence (given by the pair (g, ι)), there exists a subsequence (given by ι_0), an element of A (given by a), and a constant $c \in (0, \infty)$ such that, up to c, the sequence a gives the asympthotic behaviour of g along a subsequence ι_0. We interpret this result in two ways:

- the set A is a set of test functions that verifies that $\mathcal{R}^{>0}$ satisfies the following "compactification property": "every sequence have a convergent partial subsequence".
- the set of test functions A "captures" every possible asympthotic behaviour.

We can impose additional restrictions on A and on the set of sequences considered.

Theorem 2. *Assume the continuum hypothesis. Let $f_0, f_1 : \omega \to \mathbb{R}^{>0}$ such that $\frac{f_0(i)}{f_1(i)} < \frac{f_0(i-1)}{f_1(i-1)}$ for $i > 0$. Let $\mathcal{R}^{>0}(f_0, f_1) = \{f \mid f : \omega \to \mathbb{R}^{>0}, f(i) \in [f_0(i), f_1(i)]\}$ the set of positive real sequences between f_0 and f_1. There exists a set $A \subset \mathcal{R}^{>0}(f_0, f_1)$ such that 1 and 2 in Theorem 1 are satisfied and, moreover, $f_0, f_1 \in A$.*

Applications to limits of graphs. Let $\mathcal{G}^\circ$ be the set of finite graphs with one loop in each vertex, and $\mathcal{G}$ the set of finite graphs. For each $F \in \mathcal{G}$, let A_F denote a set of sequences of positive real numbers obtained by using Theorem 2 with $f_0(n) = n$ and $f_1(n) = n^{|V(F)|}$, and with both sequences in A_F, where $V(F)$ is the vertex set of F. Note that, for each $G \in \mathcal{G}^\circ$, $|\{h : F \to G \,:\, h \text{ is a graph homomorphism}\}| \in [n, n^{|V(F)|}]$.

Let $\{G_i\}_{i\in I}$ be a sequence of graphs in $\mathcal{G}^\circ$ with strictly increasing number of vertices (not necessarily $|G_i| = i$, just an strictly increasing sequence). We say that (see [12, Definition 2.1])

$$\{G_i\}_{i\in I} \text{ is } q\text{-convergent to } \{(f_F, c_F)\}_{f_F\in A_F, c_F\in(0,\infty)} \iff$$

$$\text{for each } F \in \mathcal{G}, \qquad \lim_{i\to\infty} \frac{|\{h : F \to G_i \,:\, h \text{ is a graph homomorphism}\}|}{f_F(\,|V(G_i)|\,)} = c_F \tag{1}$$

Note that the use of $\mathcal{G}^\circ$ instead of $\mathcal{G}$ is mostly for technical reasons, as we always want to consider sequences of non-zero real numbers. Note also that, by doing inclusion–exclusion arguments, the number of homomorphisms from F to G' (with the loops removed) can be recovered from the number of homomorphisms from F_i to G (graph with one loop on each vertex), where $\{F_i\}$ are the subgraphs of G. Now, a couple of results that gives the application of the "compactification" result to limits of graphs.

Proposition 3. *Assume the continuum hypothesis. Let $\{G_i\}_{i\in I}$ is an infinite sequence of graphs, with strictly increasing number of vertices, then there exists an infinite $I_0 \subseteq I$ such that $\{G_i\}_{i\in I_0}$ is q-convergent.*

Equivalently, any sequence has a partial convergent subsequence.

Proposition 4. *Let $\{G_i\}_{i\in I}$ is an infinite sequence of graphs in $\mathcal{G}^\circ$, with strictly increasing number of vertices, and such that for each $F \in \mathcal{G}$,*

$$\lim_{i\to\infty} |\{h : F \to G_i \,:\, h \textit{ is a graph homomorphism}\}| \,/\, \left[|V(G_i)|^{|V(F)|}\right] = c_F,\ c_F > 0$$

then $\{G_i\}_{i\in I}$ is also q-convergent.

Equivalently, if the sequence is convergent in the dense sense with positive probabilities ([9]), then it is also q-convergent using the same constants and functions $n \to n^{|V(F)|}$ for each $F \in \mathcal{G}$. In this case, the same would be true for the sequences of graphs where the loops have been removed.

Proposition 5. *Let $\{G_i\}_{i\in I}$ be an infinite sequence of graphs each of which has maximum degree upper bounded by d, belong to $\mathcal{G}^\circ$, and the number of vertices is strictly increasing along the sequence. Assume that, for each graph $F \in \mathcal{G}$ we have*

$$\lim_{i\to\infty} |\{h : F \to G_i \,:\, h \text{ is an graph homomorphism}\}| \,/|V(F)| = c_F,\ c_F > 0$$

if and only if $\{G_i\}_{i\in I}$ is q-convergent.

Note that the fact that $\{G_i\}_{i\in I}$ is a sequence of bounded degree graphs we may ask whether it converges in the Benjamini-Schramm sense [2]; in this case, by an inclusion-exclusion argument, the convergence considered in Proposition 5 is equivalent to the local-convergent considered by Benjamini-Schramm [2]. In this case, the loops ensures a bare minimum of homomorphisms for each subgraph. Therefore, Proposition 5 claims that, for bounded-degree graph sequences, Benjamini-Schramm convergence is equivalent to q-convergent.

Altoghether, Proposition 5 and Proposition 4 shows that the notion of q-convergence extends the notion of convergence for the limits of graphs in the dense case [9], and the notion of convergence for in the case of sequences of bounded degree graphs considered by Benjamini and Schramm [2] into a single, uniform framework. Here we should note that asking for the constants $c_F > 0$ in the dense case Proposition 4 is rather natural, as there are many sequences that are convergent in the dense case and that, for instance, have *no* triangles (or copies of K_3), to the same limit, yet there are several subsequences with different growth ratios of triangles, and thus the q-convergence will distinguish between the two subsequences. The q-convergence may distinguish different sequences that the dense notion considers to be equivalent; this is rather a natural behaviour since we want to distinguish between sparse sequences that the dense notion of convergence maps to the zero graphon [9]. The presence of Proposition 3 allows to claim reasonable compactification properties for the set of q-convergent sequences of graphs.

2 Sketch of the arguments

Let us sketch the proof of Theorem 1 (the key difference in Theorem 2 is highlighted below). The (positive) real numbers $\mathbb{R}^{>0}$ have the cardinality of the continuum, the set of sequences of positive reals numbers $\prod_{i\in\omega}\mathbb{R}^{>0}$ has the cardinality of the continuum, since $|\prod_{i\in\omega}\mathbb{R}^{>0}| = (\mathfrak{c})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}$, and the set of ordered injections $\mathcal{I} = \{\iota \mid \iota : \omega \to \omega \text{ ordered injection}\}$ has the cardinality of the continuum as well (note that these injections are a subset of all possible subsets of the natural numbers). Therefore, $[\prod_{i\in\omega}\mathbb{R}^{>0}] \times \mathcal{I}$ the set of infinite subsequences of positive real numbers has the cardinality of the continuum. Assuming the continuum hypothesis and the axiom of choice, we can well-order $[\prod_{i\in\omega}\mathbb{R}^{>0}]\times\mathcal{I}$ and biject it with the countable ordinals (those ordinals $<\omega_1$). Thus we write $[\prod_{i\in\omega}\mathbb{R}^{>0}]\times\mathcal{I}=\{(f_\alpha,\iota_\alpha)\mid\alpha<\omega_1\}$ and use a transfinite induction along ω_1 to find sets $\{A_\alpha\,|\,\alpha\le\omega_1\}$ that have the desired properties (the sequences of A_α are pairwise comparable, and each subsequence (f_β,ι_β), with $\beta<\alpha$ has a representative in A_α). The sets A_α are built as $A_0=\emptyset$ (or $A_0=\{f_0,f_1\}$ if we want to show Theorem 2), $A_\alpha=\cup_{\beta<\alpha}A_\beta$ for limit ordinals, and where $A_{\alpha+1}$ is build from A_α by adding a new sequence agreeing uppon (f_α,ι_α) along a subsequence and comparable with the others in A_α; to find this new sequence we first examine whether there is already a test sequence in A_α that agrees with (f_α,ι_α) along a subsequence (up to a multiplicative c), if that is the case, then $A_{\alpha+1}=A_\alpha$. If that is not the case, then we can partition the at most countable (here we are using the continuum hypothesis again) elements in A_α in two parts U and D, and find a sequence of subsequences $\{\iota_{\alpha,i}\}_{i<\omega}$ ($\iota_{\alpha,i}$ subsequence of $\iota_{\alpha,i+1}$) for (f_α,ι_α) such that (f_α,ι_α) is below g_i along $\iota_{\alpha,i}$ if $g_i\in A_\alpha$ belongs to U, and is above g_j along $\iota_{\alpha,j}$ if $g_j\in A_\alpha$ belongs to D. Then we find a subsequence $\iota_{\alpha,\omega}$ of (f_α,ι_α) along which (f_α,ι_α) is below each element from A_α in U and above each element in D. Finally, we complete the subsequence along $\iota_{\alpha,\omega}$ into a full sequence between the elements of U and the elments of D by backwards extending the elements along the subsequence $\iota_{\alpha,\omega}$ with elements between the lowest found elements in U and the highest found elements in D (the current element of the subsequence $\iota_{\alpha,\omega}$ ensures that the multiplicative distance to all the previously considered elements in U and D goes to 0 and ∞ respectively). The transfinite induction then gives A_{ω_1}.

Proposition 3 is proven finding an appropriate triple (subsequence, constant, test function) for each finite graph. These test functions capture the asympthotic growth f_F for each subgraph F, consistently along an increasing family of subgraphs F by considering further subsequences of the graphs $\{G_i\}$. Then we use a diagonal argument, similar as before, to find a subsequence of graphs $\{G_i\}$ that, for each F, the number of homomorphisms from F to $\{G_i\}$ has, along the subsequence of graphs $\{G_i\}$, the asympthotic growth f_F up to a multiplicative constant c_F.

Proposition 5 and Proposition 4 follows by observing that, with their respective hypotheses, the test functions at which the subgraph count of a convergent graph sequence grow are, up to a multiplicative factor, the minimum and maximum possible (given that the graphs have loops on each vertex).

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