Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications
EUROCOMB'23
Prague, August 28 - September 1, 2023

Crux, space constraints and subdivisions

(Extended abstract)

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Abstract

The existence of H-subdivisions within a graph G has deep connections with topological, structural and extremal properties of G. One prominent example of such a connection, due to Bollobás and Thomason and independently Komlós and Szemerédi, asserts that the average degree of G being d ensures a $K_{\Omega(\sqrt{d})}$-subdivision in G. Although this square-root bound is the best possible, various results showed that much larger clique subdivisions can be found in a graph for many natural classes. We investigate the connection between crux, a notion capturing the essential order of a graph, and the existence of large clique subdivisions.

Our main result gives an asymptotically optimal bound on the size of a largest clique subdivision in a generic graph G, which is determined by both its average degree and its crux size. As corollaries, we obtain

- a characterisation of extremal graphs for which the square-root bound above is tight: they are essentially a disjoint union of graphs each of which has the crux size linear in d;
- a unifying approach to find a clique subdivision of almost optimal size in graphs which do not contain a fixed bipartite graph as a subgraph;

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Supported by Institute for Basic Science (IBS-R029-C4), the POSCO Science Fellowship of POSCO TJ Park Foundation, and the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) No. RS-2023-00210430.

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Supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIT) No. RS-2023-00210430.

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- and that the clique subdivision size in random graphs $G(n, p)$ witnesses a dichotomy: when $p = \omega(n^{-1/2})$, the barrier is the space, while when $p = o(n^{-1/2})$, the bottleneck is the density.

DOI: `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-084`

1 Introduction

For a graph H, a subdivision of H (or an H-subdivision) is a graph obtained by replacing each edge of H by internally vertex-disjoint paths. Studies on the existence of certain subdivisions in a given graph G provide deep understandings on various aspects of G. For example, the cornerstone theorem of Kuratowski [12] in 1930 completely characterises planar graphs by proving that graphs are planar if and only if they do not contain a subdivision of either K_5, the complete graph on five vertices, or $K_{3,3}$, the complete bipartite graph with three vertices in each class.

What conditions on graphs G guarantee an H-subdivision in them? A fundamental result of Mader [14] in 1967 states that a large enough average degree always provides a desired subdivision. Namely, for every $t \in \mathbb{N}$, there exists a smallest integer $f(t)$ such that every graph G with average degree at least $f(t)$ contains a subdivision of K_t. He further conjectured that $f(t) = O(t^2)$. This conjecture was verified in the 90s by Bollobás and Thomason [3] and independently by Komlós and Szemerédi [10]. In fact, $f(t) = \Theta(t^2)$; the lower bound was observed by Jung [8] in 1970: consider the n-vertex graph which is a disjoint union of $n/5t^2$ copies of $K_{\frac{t^2}{10},\frac{t^2}{10}}$. A clique subdivision must be embedded in a connected graph; this example, though may have arbitrary large order n, is essentially the same as one copy of $K_{\frac{t^2}{10},\frac{t^2}{10}}$, which does not contain a K_t-subdivision. Indeed, at least $\binom{t/2}{2}$ many edges are subdivided in any K_t-subdivision in $K_{\frac{t^2}{10},\frac{t^2}{10}}$, which would require around $t^2/8 > t^2/10$ vertices on one side. In other words, apart from the obvious "degree constraint" from the average degree, there is also some "space constraint" forbidding a K_t-subdivision.

From the extremal example above, it is then natural to wonder if G does not structurally look like $K_{\frac{t^2}{10},\frac{t^2}{10}}$, can we find a larger clique subdivision? Indeed, Mader [15] conjectured that every C_4-free graph with average degree d contains a subdivision of $K_{\Omega(d)}$ and recently it was resolved by Liu and Montgomery [13]. Furthermore, they proved that for every $t \geq s \geq 2$, there exists a constant $c = c(s, t)$ such that if G is $K_{s,t}$-free and has average degree d, then G has a subdivision of a clique of order $cd^{s/2(s-1)}$.

Note that a C_4-free graph with average degree at least d must have at least $\Omega(d^2)$ vertices as the maximum number of edges of an n-vertex C_4-free graph is $O(n^{3/2})$, hence providing enough space to put a $K_{\Omega(d)}$-subdivision with $O(d^2)$-vertices. Similarly, the number $d^{s/2(s-1)}$ also matches with the conjectured extremal number of $K_{s,t}$. Thus, all these H-free conditions relax the "space constraints". Hence, this suggests that 'the essential order' of the graph G, rather than structural F-freeness, is an important factor for the size of the largest clique subdivision. Indeed, Liu and Montgomery [13] conjectured that every graph G with its 'essential order' n and average degree d contains a K_t subdivision with $t = \Omega(\min\{d, \frac{n}{\log n}\})$.

Such a notion of 'essential order', called crux, was recently introduced by Haslegrave, Hu, Kim, Liu, Luan and Wang [5]. We write $d(G)$ for the average degree of G.

Definition 1.1 (Crux)**.** Let $\alpha > 0$ and G be a graph. A subgraph $H \subseteq G$ is an *α-crux* if $d(H) \geq \alpha \cdot d(G)$. Let $c_\alpha(G)$ be the order of a smallest α-crux in G, that is:

$$c_\alpha(G) = \min\{n : \exists H \subseteq G \text{ s.t. } |H| = n, d(H) \geq \alpha d(G)\}.$$

We will write simply $c(G)$ when $\alpha = 1/100$; the choice of $1/100$ here is not special and can be replaced with any small number. Roughly speaking, the crux of a graph is large when the edges are relatively uniformly distributed.

Our main result reads as follows. It implies in particular that the space constraints, measured by the crux size, is a deciding factor for the size of largest clique subdivision in a graph.

Theorem 1.2. *There exists an absolute constant $\beta > 0$ such that the following is true. Let G be a graph with $d(G) = d$. Then G contains a $K_{\beta t/(\log\log t)^6}$-subdivision where*

$$t = \min\left\{d, \sqrt{\frac{c(G)}{\log c(G)}}\right\}.$$

Theorem 1.2 asymptotically confirms a conjecture of Liu and Montgomery [13]. The bound above is optimal up to the multiplicative $(\log\log t)^6$ factor: the d-blowup of a d-vertex $O(1)$-regular expander satisfies $c(G) = \Theta(d^2)$ and the largest clique subdivision has order $d/\sqrt{\log d}$ (see [13] for more details).

2 Applications

2.1 Characterisation of extremal graphs

The first consequence of our main result is a structural characterisation of extremal graphs G having the smallest possible clique subdivision size $\Theta(\sqrt{d(G)})$, showing that the *only obstruction* to get a larger than usual clique subdivision is a small crux. In other words, if $c(G) = \omega(d)$, then one can embed a K_t-subdivision with $t = \omega(\sqrt{d})$. Theorem 1.2 does not imply this result but along the way of proving it, we obtain this.

Theorem 2.1. *Given a graph G with average degree d, if the largest clique subdivision has order $\Theta(\sqrt{d})$, then its crux size is linear in d, i.e. $c(G) = O(d)$.*

Theorem 2.1 implies that the extremal graphs are *essentially disjoint union of dense small graphs whose crux size is linear in their average degrees.* This can be viewed as an analogous result of Myers [16] who studied the extremal graphs for embedding clique minors.

2.2 Graphs without a fixed bipartite graph

The next application provides a lowerbound on the largest clique subdivision size, which is optimal up to a polylog-factor, in a graph without a fixed copy of bipartite graph H. This generalises the result of Liu and Montgomery [13] on $K_{s,t}$-free graphs. We would like to remark that the proof of Liu and Montgomery makes heavy use of the structure of the forbidden graph $K_{s,t}$, hence their argument does not extend to general H-free graphs. Below, we write $x = \tilde{\Omega}(y)$ if there exists positive constants a, b such that $x \geq ay\log^{-b} y$.

Corollary 2.2. *Let H be a bipartite graph with extremal number $\mathrm{ex}(n, H) = O(n^{1+\tau})$ for some $0 < \tau < 1$ and let G be an H-free graph with average degree d. Then G contains a K_t-subdivision where*

$$t = \begin{cases} \tilde{\Omega}(d^{\frac{1}{2\tau}}) & \text{if } \tau > 1/2 \\ \tilde{\Omega}(d) & \text{if } \tau \leq 1/2. \end{cases}$$

Proof. Let $\alpha = 1/100$, and F be a smallest α-crux of G of order $c(G)$. As F is H-free, $e(F) \leq O(|F|^{1+\tau})$, hence

$$c(G) = |F| = \Omega(d(F)^{1/\tau}) = \Omega(d(G)^{1/\tau}).$$

Therefore, Theorem 1.2 implies that G has a clique subdivision of size $\tilde{\Omega}(\min\{d, c(G)^{\frac{1}{2}}\}) = \tilde{\Omega}(\min\{d, d^{\frac{1}{2\tau}}\})$. □

The bound above is best possible up to the polylogarithmic factor if $\mathrm{ex}(n, H) = \Theta(n^{1+\tau})$. To see this, let G be an n-vertex bipartite H-free graph G with $\Theta(n^{1+\tau})$ edges. If G has a K_t-subdivision, then at least $t/2$ core vertices are in the same part of a bipartition of G. For any two of them, a path connecting them uses at least one vertex of the other part of G so $\binom{t/2}{2} \leq n$ and therefore $t = O(\sqrt{n}) = O(d^{1/2\tau})$.

2.3 Dichotomy on Erdős-Rényi random graphs

The last application delas with the subdivisions in Erdős-Rényi random graphs. While the size of the largest K_t-minor in a random graph is widely studied (for example [2, 11, 4]), the only known results for clique subdivision is when p is a constant. More precisely, when $p \in (0, 1)$ is a constant, Bollobás and Catlin [1] proved in 1981 that the largest clique subdivision of $G(n, p)$ is $(\sqrt{2/(1-p)} + o(1))\sqrt{n}$ with high probability (w.h.p.).

We determine the size of the largest clique subdivision upto polylog-factor when $p = \omega(\frac{\log n}{n})$. We remark that when $p = o(\frac{\log n}{n})$, the clique subdivision is typically extremely small in $G(n, p)$: only logarithmic in n.

Corollary 2.3. *Suppose $p = \omega(\frac{\log n}{n})$ and $p = 1 - \Omega(1)$. Then w.h.p., the largest t that $G = G(n, p)$ has a K_t-subdivision is given by*

$$t = \tilde{\Theta}(\min\{np, \sqrt{n}\}).$$

The proof is obtained by showing $c(G) = \Omega(n)$ for a binomial ramdom graph using the standard concentration inequalities. Corollary 2.3 implies an interesting dichotomy on clique subdivision size in $G(n,p)$ above and below the density $1/\sqrt{n}$: when $p = \omega(n^{-1/2})$, then it is limited solely by the space constraints, while when $p = o(n^{-1/2})$, the degree constraint is the bottleneck.

3 Outline of the proof

In this section, we sketch the proof of our main theorem. For the detail version of the proof, see the online vertsion of our preprint [7].

3.1 Sublinear expander

A main tool we use in this paper is the sublinear expander notion introduced by Komlós and Szemerédi [9, 10]. Let $N^i_G(X)$ be the set of vertices which is distance exactly i from X. In particular, $N^0_G(X) = X$. We write $N_G(X)$ to denote $N^1_G(X)$ and we write $B^i_G(X) = \bigcup_{j\le i} N^i_G(X)$. For given ε, k, we define $\rho(x) = \rho(x,\varepsilon,k)$ as

$$\rho(x) = \begin{cases} 0 & \text{if } x < \frac{k}{5} \\ \frac{\varepsilon}{\log^2(15x/k)} & \text{if } x \ge \frac{k}{5} \end{cases}$$

Note that $\rho(x)$ is a decreasing function and $x\rho(x)$ is a increasing function for $x \ge \frac{k}{2}$. Komlós and Szemerédi introduced the notion of (ε,k)-expander, which is a graph in which every set of appropriate size has not too small external neighbourhood. Haslegrave, Kim and Liu [6] slightly generalised this notion to a robust version. Roughly speaking, a graph G is a robust-expander if every set of appropriate size has not too small external neighbourhood even after deleting a small number of vertices and edges. For an edge set $F \subseteq E(G)$, we write $G \setminus F$ to denote the graph with the vertex set $V(G)$ and the edge set $E(G) \setminus F$.

Definition 3.1 ([6])**.** For $\varepsilon > 0, k > 0$, a graph G is (ε,k)*-robust-expander* if for every subset $X \subseteq V(G)$ of size $\frac{k}{2} \le |X| \le \frac{|V(G)|}{2}$ and an edge set $F \subseteq E(G)$ with $|F| \le d(G)\rho(|X|)|X|$, we have $|N_{G-F}(X)| \ge \rho(|X|)|X|$.

This notion of sublinear expander is very useful in the following three aspects.

- Every graph contains a robust-expander subgraph with almost the same average degree.
- This provides a short connection between any two large sets while avoiding a relatively small set of vertices and edges.
- No metter which small set of vertices we delete, the remaining graph still has large average degree.

These three aspects are captured in the following three results.

Theorem 3.2. *[6, 9, 10] If $\varepsilon > 0$ is sufficiently small (independent from k) so that $\int_1^\infty \rho(x,\varepsilon,k)/x dx < \frac{1}{8}$, then every graph G contains an (ε,k)-robust-expander H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$ as a subgraph.*

Lemma 3.3. *[6, 10] Let G be an n-vertex (ε,k)-robust-expander. Then for any two vertex sets X_1, X_2 of size at least $x \geq \frac{k}{2}$, and a vertex set W of size at most $\frac{x\rho(x)}{4}$, there exists a path between X_1 and X_2 in $G - W$ of length at most $\frac{2}{\varepsilon}\log^3(\frac{15n}{k})$.*

Lemma 3.4. *Suppose $0 < \frac{1}{n} \ll \varepsilon < 1$ and $k < \frac{n}{10}$. Let G be an n-vertex (ε,k)-robust-expander. Then for every $W \subseteq V(G)$ with $|W| \leq \frac{1}{20}\rho(n,\varepsilon,k)\cdot n$, we have $d(G-W) \geq \frac{1}{20}\rho(n,\varepsilon,k)\cdot d(G)$.*

3.2 Outlines of the main steps

Assume that G and t are as in Theorem 1.2 and let $s = \beta t/(\log\log t)^6$ for some small enough constant $\beta > 0$. Using Lemma 3.2, we can assume that our graph is an expander with an appropriate choice of ε and k. Our proof bacisally use the structures introduced in [13]. We find $\Omega(s)$ rooted trees with many branches and leaves (called units and webs) and build short vertex-disjoint paths between them to form a desired clique subdivision. On the other hand, directly mimicking such an argument does not provide a subdivision of the desired size. We need to find a clique subdivision of the size which almost matches the smaller bounds of the degree constraint d and the space constraint $\sqrt{c(G)/\log c(G)}$.

If the bound from the degree constraint is stronger (i.e., d is smaller than $\sqrt{c(G)/\log c(G)}$), the main goal is to collect many vertices of degree at least t. If the bound from the space constraint is stronger (i.e., $\sqrt{n/\log n}$ is much smaller than d), the main difficulty is to find short paths connecting the vertices. As we want to build $\Omega(s^2)$ vertex disjoint paths in G to form a K_s-subdivision, we need to be able to find $\Omega(s^2)$ paths of average length at most $O(n/s^2) = \log n(\log\log n)^{O(1)}$. However, Lemma 3.3 only guarantees a much longer path of length $O(\log^3(n/k))$. To obtain paths of desired length, we may take k to be $n/(\log n)^{O(1)}$, but such a choice of k does not allow us to obtain expansions of small sets, introducing another difficulty. Another issue from this approach is that we don't have any controls on the order of (ε,k)-expander we take. The order of such an expander puts additional space constraints for finding a desired subdivision as well as affect the length of paths we obtain from Lemma 3.3.

To overcome the above difficulties, we consider several cases and use (ε,k)-expander in each case with different choices of k. Let $d = d(G)$. By iteratively applying Theorem 3.2, we obtain the following graphs

$$G \supseteq G_1 \supseteq G_2 \supseteq H$$

where G_1 is an $(\varepsilon,\varepsilon d)$-expander, G_2 is an (ε,d^2)-expander, and H is an $(\varepsilon,c(G)/100)$-expander such that each graph has minimum degree at least $d/16$. Let $n_1 = |G_1|, n_2 = |G_2|$, and $n_H = |H|$. We now consider the following four cases depending on the values of t, n_1, n_2, and n_H.

Case 1: $\boldsymbol{d \le \exp(\log^{1/6} n_1)}$ **:** In this case, we can adapt a theorem in [13] to obtain a desired $K_{\Omega(d)}$-subdivision.
Case 2: $\boldsymbol{t \le \min\{\frac{\sqrt{n_1}}{(\log n_1)^{O(1)}}, \frac{d}{(\log d)^{O(1)}}\}}$ **:** In this case, as t is quite smaller than both d and $\sqrt{n_1/\log n_1}$, the degree constraint and the space constraint within G_1 are not strong obstacles to obtain a desired $K_{\Omega(t)}$-subdivision. Hence, by utilizing the properties of the expanders, we can construct many units and webs and connect them with short paths to obtain a desired $K_{\Omega(t)}$-subdivision.
Case 3: $\boldsymbol{d \le \frac{\sqrt{n_2}}{(\log n_2)^{O(1)}}}$**:** In this case, the space constraint within G_2 is much weaker than the degree constraint, so our main concern is to collect $\Omega(t)$ vertices of degree at least $d(\log n_2)^{O(1)}$. Although Lemma 3.4 provides a set of large degree vertices, it only gives vertices of degree $d/(\log n_2)^{O(1)}$, which is smaller than what we need. Considering two cases where the edge distribution of G_2 is close to uniform and skewed, a careful analysis provides a desired set of vertices of large degree in both cases.
Case 4: The remaining cases: In the remaining case, we will find a desired K_s-subdivision in H. Note that as we are not in case 1–3, we obtain the ineqaulity

$$\frac{n_H}{(\log c(G))^{O(1)}} \le c(G) \le n_H.$$

As H is $(\varepsilon, c(G)/100)$-expander, this ensures that $\rho(x, \varepsilon, c(G)/100) = (\log\log c(G))^{O(1)}$ for every $c(G) \le x \le n_H$. With this extra assumption on our hand, Lemma 3.4 now provides a set of $\Omega(t/(\log\log t)^{O(1)})$ vertices of degree $\Omega(t/(\log\log t)^{O(1)})$, which matches the bound from the degree constraint. Moreover, Lemma 3.3 also yields a path of length $(\log\log c(G))^{O(1)}$ between two large sets of size at least $c(G)/100$. Note that the definition of crux ensures some expansion of all vertex set smaller than $c(G)/100$. By utilizing this, we can show that the $O(\log c(G))$-th ball $B_H^{O(\log c(G))}(v)$ of a well-chosen vertex v has size at least $c(G)/100$. This together with Lemma 3.3 provides a desired path of length $(\log\log c(G))^{O(1)}$ between two balls $B_H^{O(\log c(G))}(v)$ and $B_H^{O(\log c(G))}(u)$ of well-chosen vertices u and v. Combining these ideas with further technical analysis, we obtain the desired K_s-subdivison. We omit further details.

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